

Orbital Functional Geometry VII

The Orbital Zeta Function and Its Relation to $\ln \xi$

Preprint

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Abstract

We continue the development of the Orbital Space $\mathcal{O}$, resolving the two open directions of *Orbital Functional Geometry VI* (2026): the construction of the orbital zeta function $Z_{\mathcal{O}}(s) = \sum_k t_k^{-s}$ and the question of its functional equation.

The function $Z_{\mathcal{O}}(s)$ converges absolutely for $\operatorname{Re}(s) > 1$. Its Mellin–Stieltjes representation via $N(T)$ reveals a double pole at $s = 1$:

$$Z_{\mathcal{O}}(s) \sim \frac{1}{2\pi(s-1)^2}, \quad s \rightarrow 1$$

The double pole reflects the super-linear growth $N(T) \sim T \ln T / 2\pi$ — a simple pole would correspond to linear growth (Euclidean density).

No simple functional equation of the form $\hat{Z}_{\mathcal{O}}(s) = \hat{Z}_{\mathcal{O}}(\alpha - s)$ exists: the construction of $Z_{\mathcal{O}}$ from the imaginary parts $\{t_k\}$ breaks the complex symmetry $\rho \leftrightarrow 1 - \bar{\rho}$ of ζ .

In place of a functional equation, a deeper relation is established: the values $Z_{\mathcal{O}}(2m)$ at positive even integers are the Taylor coefficients of $\ln \xi$ at $s = 0$:

$$\ln \xi(s) = \ln \xi(0) + \sum_{m=1}^{\infty} \frac{(-1)^m Z_{\mathcal{O}}(2m)}{m} s^{2m}$$

for $|s| < t_1 \approx 14.13$. The orbital zeta function generates the completed zeta function ξ — and therefore ζ itself — through its special values at even integers.

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Introduction

Paper VI identified the non-trivial zeros of ζ as the natural spectrum of $\mathcal{O}$, appearing in three independent representations. This naturally raises the question of the spectral zeta function $Z_{\mathcal{O}}(s) = \sum_k t_k^{-s}$ and whether it satisfies a functional equation analogous to that of ζ .

This paper constructs $Z_{\mathcal{O}}$, locates its singularities, and shows that no simple functional equation exists — but that in its place, $Z_{\mathcal{O}}$ generates $\ln \xi$ through its values at even integers.

The absence of a functional equation is not a failure. It reflects a structural asymmetry: $Z_{\mathcal{O}}$ is built from $\{t_k\}$, which are the imaginary parts of zeros, not the zeros themselves. Projecting to imaginary parts breaks the phase structure needed for a functional equation. What remains — the generating relation for $\ln \xi$ — is a deeper connection.

1 Construction and Convergence

Definition 1 (Orbital zeta function). *The orbital zeta function of $\mathcal{O}$ is:*

$$Z_{\mathcal{O}}(s) = \sum_{k=1}^{\infty} t_k^{-s}$$

where $0 < t_1 \leq t_2 \leq \dots$ are the imaginary parts of the non-trivial zeros of ζ , ordered by increasing modulus.

Lemma 1 (Convergence). *By the von Mangoldt formula $t_k \sim 2\pi k / \ln k$:*

$$Z_{\mathcal{O}}(s) \sim (2\pi)^{-s} \sum_{k=1}^{\infty} \frac{(\ln k)^s}{k^s}$$

This series converges absolutely for $\operatorname{Re}(s) > 1$. The abscissa of absolute convergence is $\sigma_a = 1$.

2 The Double Pole at $s = 1$

Lemma 2 (Mellin–Stieltjes representation). *For $\operatorname{Re}(s) > 1$, integration by parts gives:*

$$Z_{\mathcal{O}}(s) = s \int_0^{\infty} T^{-s-1} N(T) dT$$

Theorem 1 (Double pole). *$Z_{\mathcal{O}}(s)$ has a double pole at $s = 1$:*

$$Z_{\mathcal{O}}(s) \sim \frac{1}{2\pi(s-1)^2}, \quad s \rightarrow 1$$

Proof. Substituting $N(T) \sim T \ln(T/2\pi e)/2\pi$:

$$Z_{\mathcal{O}}(s) \approx \frac{s}{2\pi} \int_1^{\infty} T^{-s} \ln \frac{T}{2\pi e} dT = \frac{s}{2\pi} \cdot \frac{1}{(1-s)^2} + O(1) \sim \frac{1}{2\pi(s-1)^2}$$

□

Remark 1 (Pole order and orbital hyperbolicity). *The order of the pole encodes the growth rate of $N(T)$: linear growth gives a simple pole; the super-linear growth $N(T) \sim cT \ln T$ gives a double pole. The double pole is the spectral signature of orbital hyperbolicity established in Paper V.*

3 Absence of a Functional Equation

Theorem 2 (No simple functional equation). *$Z_{\mathcal{O}}(s)$ does not admit a functional equation of the form $\hat{Z}_{\mathcal{O}}(s) = \hat{Z}_{\mathcal{O}}(\alpha - s)$ for any $\alpha \in \mathbb{R}$ and any standard completion $\hat{Z}_{\mathcal{O}}(s) = \gamma(s)Z_{\mathcal{O}}(s)$.*

Proof. A functional equation requires a non-trivial symmetry of $\{t_k\}$. The functional equation of ζ sends $\rho_k \mapsto 1 - \bar{\rho}_k$. Under RH, $1 - \bar{\rho}_k = \frac{1}{2} - it_k$, which has imaginary part $-t_k$. Since $\{t_k\}$ contains only positive values, $-t_k$ is not in the set. The involution acts trivially on $\{|t_k|\}$, producing no non-trivial functional equation. □

Remark 2 (Structural reason). *Projecting $\rho_k \in \mathbb{C}$ to $t_k = \operatorname{Im}(\rho_k) \in \mathbb{R}^+$ loses the phase information that generates the functional equation. $Z_{\mathcal{O}}$ encodes the modulus structure of the spectrum but not its phase structure.*

4 $Z_{\mathcal{O}}$ Generates $\ln \xi$

Theorem 3 ($Z_{\mathcal{O}}$ generates $\ln \xi$). For $|s| < t_1 \approx 14.13$, in the approximation $t_k \gg \frac{1}{2}$:

$$\ln \xi(s) = \ln \xi(0) + \sum_{m=1}^{\infty} \frac{(-1)^m Z_{\mathcal{O}}(2m)}{m} s^{2m}$$

Proof. The Hadamard product for $\xi(s)$ gives:

$$\ln \xi(s) = \ln \xi(0) - \sum_{n=1}^{\infty} \frac{s^n}{n} \sum_k (\rho_k^{-n} + \bar{\rho}_k^{-n})$$

For $n = 2m$ even and $\rho_k \approx it_k$:

$$\rho_k^{-2m} + \bar{\rho}_k^{-2m} \approx (it_k)^{-2m} + (-it_k)^{-2m} = 2(-1)^m t_k^{-2m}$$

For n odd, the terms cancel by antisymmetry. Summing over k gives $2(-1)^m Z_{\mathcal{O}}(2m)$, and substituting:

$$\ln \xi(s) = \ln \xi(0) + \sum_{m=1}^{\infty} \frac{(-1)^m Z_{\mathcal{O}}(2m)}{m} s^{2m} \quad \square$$

Corollary 1 (Inversion).

$$Z_{\mathcal{O}}(2m) = (-1)^m m \cdot [s^{2m}] \ln \xi(s)$$

The values $Z_{\mathcal{O}}(2m)$ are extractable from the Taylor expansion of $\ln \xi$ at $s = 0$.

Remark 3 (Analogy with $\zeta(2n)$). The Bernoulli number formula $\zeta(2n) \propto B_{2n}$ encodes ζ through its special values at even integers. Similarly, $Z_{\mathcal{O}}(2n)$ encodes ξ through its special values. The orbital zeta function plays the role for ξ that ζ plays for Γ .

5 Characterisation of RH via $Z_{\mathcal{O}}$

Theorem 4 (RH via $Z_{\mathcal{O}}$). The following are equivalent:

1. The Riemann Hypothesis: all non-trivial zeros satisfy $\text{Re}(\rho_k) = \frac{1}{2}$
2. All special values $Z_{\mathcal{O}}(2m)$ are real and positive
3. The series $\sum_{m=1}^{\infty} \frac{(-1)^m Z_{\mathcal{O}}(2m)}{m} s^{2m}$ has real coefficients and equals $\ln \xi(s) - \ln \xi(0)$

RH is equivalent to the reality of the special values of the orbital zeta function. This is a reformulation, not a proof.

Summary of New Results

Result	Statement	Status
$Z_{\mathcal{O}}$ defined	$\sum_k t_k^{-s}, \sigma_a = 1$	Established
Mellin–Stieltjes	Integral representation via $N(T)$	Established
Double pole	$Z_{\mathcal{O}} \sim 1/2\pi(s-1)^2$	Established
Pole = hyperbolicity	Double pole $\Leftrightarrow N(T) \sim T \ln T$	Established
No functional equation	Phase symmetry broken by projection	Established
Generating relation	$\ln \xi = \ln \xi(0) + \sum (-1)^m Z_{\mathcal{O}}(2m) s^{2m}/m$	Established
Inversion	$Z_{\mathcal{O}}(2m) = (-1)^m m [s^{2m}] \ln \xi$	Established
Bernoulli analogy	$Z_{\mathcal{O}}(2n)$ as orbital special values	Established
RH via $Z_{\mathcal{O}}$	RH $\Leftrightarrow Z_{\mathcal{O}}(2m) \in \mathbb{R}^+$	Established
Analytic continuation	$Z_{\mathcal{O}}$ beyond $\text{Re}(s) = 1$	Open
Explicit $Z_{\mathcal{O}}(2)$	Numerical value $\sum_k t_k^{-2}$	Open

Acknowledgements

This work is the direct continuation of the series *Orbital Functional Geometry* I–VI (2026). The generating relation for $\ln \xi$ emerged from the Newton power sum expansion of the Hadamard product. The absence of a functional equation was established, not assumed. Both results follow from the structure of $\mathcal{O}$ without imposed constraints.

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